

ZETA FUNCTIONS AND TOPOLOGICAL ENTROPY OF THE MARKOV-DYCK SHIFTS

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ABSTRACT. The Markov-Dyck shifts arise from finite directed graphs. An expression for the zeta function of a Markov-Dyck shift is given. The derivation of this expression is based on a formula in Keller (G. Keller, *Circular codes, loop counting, and zeta-functions*, J. Combinatorial Theory **56** (1991), pp. 75–83). For a class of examples that includes the Fibonacci-Dyck shift the zeta functions and topological entropy are determined.

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1. INTRODUCTION

Let Σ be a finite alphabet, and let S_Σ be the left shift on $\Sigma^\mathbb{Z}$,

$$S_\Sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.$$

The closed shift-invariant subsystems of the shifts S_Σ are called subshifts. For an introduction to their theory, which belongs to symbolic dynamics, we refer to [13] and [20]. A finite word in the symbols of S_Σ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears somewhere in a point of X . A subshift is uniquely determined by its language of admissible words that we denote by $\mathcal{L}(X)$. $\mathcal{L}_n(X)$ will denote the set of words in $\mathcal{L}(X)$ of length $n \in \mathbb{N}$. The topological entropy of the subshift $X \subset \Sigma^\mathbb{Z}$ is given by

$$h(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \mathcal{L}_n(X).$$

Denoting by $\Pi_n(X)$ the number of points of period n of the subshift $X \subset \Sigma^\mathbb{Z}$, the zeta function of the subshift $X \subset \Sigma^\mathbb{Z}$ is given by

$$\zeta_X(z) = e^{\sum_{n \in \mathbb{N}} \frac{\Pi_n(X) z^n}{n}}.$$

In this paper we are concerned with a class of subshifts that arise from finite directed graphs as a special case of constructions that were described in [14], [15], [10]. Following the line of terminology of [24], we call these subshifts Markov-Dyck shifts. Let G be a finite directed graph with vertex set \mathcal{V} and edge set \mathcal{E} . We denote the initial vertex of $e \in \mathcal{E}$ by $s(e)$ and the final vertex by $r(e)$. Let G^- be that graph with vertex set \mathcal{V} and edge set \mathcal{E}^- a copy of \mathcal{E} . Reverse the directions of the edges in \mathcal{E}^- to obtain the reversed graph G^+ of G^- with vertex set \mathcal{V} and edge set \mathcal{E}^+ . Denote by \mathcal{P}^- (resp. \mathcal{P}^+) the set of finite paths in G^- (resp. G^+). The mapping $e^- \rightarrow e^+$ ($e^- \in \mathcal{E}^-$) extends to the bijection $w^- \rightarrow w^+$ ($w^- \in \mathcal{P}^-$) of \mathcal{P}^- onto \mathcal{P}^+ that reverses direction. With idempotents $P_v, v \in \mathcal{V}$, the set $\mathcal{E}^- \cup \{P_v : v \in \mathcal{V}\} \cup \mathcal{E}^+$ is the generating set of the graph inverse

semigroup of G , where, besides $P_u^2 = P_u, u \in \mathcal{V}$, the relations are (see for instance [27])

$$P_u P_w = 0, \quad u, w \in \mathcal{V}, u \neq w,$$

$$f^- g^+ = \begin{cases} P_{s(f)}, & (f = g), \\ 0 & (s(f) \neq s(g), f, g \in \mathcal{E}), \end{cases} \quad (1.1)$$

and

$$g^+ f^- = 0, \quad s(f) \neq s(g),$$

$$f^- g^- = 0, \quad r(f) \neq s(g), \quad f, g \in \mathcal{E}.$$

The alphabet of the Markov-Dyck shift D_G of G is $\mathcal{E}^- \cup \mathcal{E}^+$ and a word $(e_k)_{1 \leq k \leq K}$ is admissible for D_G precisely if

$$\prod_{1 \leq k \leq K} e_k \neq 0.$$

For the directed graph with one vertex and loops $e_n, 1 \leq n \leq N, N > 1$, the relations take the form

$$e_n^- e_n^+ = \mathbf{1}, \quad 1 \leq n \leq N, \quad e_l^- e_m^+ = 0, \quad 1 \leq l, m \leq N, \quad l \neq m. \quad (1.2)$$

and one sees the Dyck inverse monoid [26], together with the Dyck shifts that were first described in [14].

The relations (1.2) can be viewed as the multiplicative relations among the relations that are satisfied by generators of a Leavitt algebra [19] or a Cuntz algebra [5] and the relations (1.1) can be viewed as the multiplicative relations among the relations that are satisfied by generators of a Leavitt path algebra of the directed graph G [1], or the generators of the graph C^* -algebra of G [6], [8].

The zeta function of the Dyck shifts were determined in [12], and K-theoretic invariants were computed in [22] and [17]. For related systems, the Motzkin shifts that add a symbol $\mathbf{1}$ to the alphabets of the Dyck shifts, the zeta functions were determined in [11] and K-theoretic invariants were computed in [21]. In section 2 we will obtain an expression for the zeta function of a Markov-Dyck shift by applying a formula of Keller's [12]. In section 3 we derive estimates for the topological entropy of the subsystems of the Markov-Dyck shifts that are obtained by allowing the paths in the Markov-Dyck shift to go from \mathcal{E}^- to \mathcal{E}^- or vice-versa only when entering a given vertex. A fortiori, this gives also estimates for the topological entropy of the Markov Dyck shifts. In section 4 we determine the zeta functions and topological entropy of the Markov-Dyck shifts that arise from directed graphs with adjacency matrix $F(a, b, c) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, $a, b, c \in \mathbb{N}$ (compare here section 10.3 of [23]). K-theoretic invariants of $D_{F(1,1,1)}$ shift were computed in [25].

The length of a word w we denote by $\ell(w)$ and we denote the generating function of a formal language \mathcal{L} by $g_{\mathcal{L}}$,

$$g_{\mathcal{L}}(z) = \sum_{n \in \mathbb{Z}_+} \text{card}\{w \in \mathcal{L} : \ell(w) = n\} z^n.$$

2. ZETA FUNCTIONS

Keller [12] has introduced the notion of a circular Markov code. Here we find ourselves in a situation where we will want a Markov code to be given by a set \mathcal{C} of non-empty words in the symbols of a finite alphabet Σ together with a finite set \mathcal{V} and mappings $r : \mathcal{C} \rightarrow \mathcal{V}, s : \mathcal{C} \rightarrow \mathcal{V}$. To (\mathcal{C}, r, s) there is associated the shift invariant set $X_{\mathcal{C}} \subset \Sigma^{\mathbb{Z}}$ of points $x \in \Sigma^{\mathbb{Z}}$ such that there are indices $I_k, k \in \mathbb{Z}$, such that

$$I_0 \leq 0 < I_1, \quad I_k < I_{k+1}, \quad k \in \mathbb{Z}, \quad (2.1)$$

and such that

$$x_{[I_k, I_{k+1})} \in \mathcal{C}, \quad k \in \mathbb{Z}, \quad (2.2)$$

and

$$r(x_{[I_k, I_{k+1})}) = s(x_{[I_{k+1}, I_{k+2})}), \quad k \in \mathbb{Z}. \quad (2.3)$$

(\mathcal{C}, r, s) is said to be a circular Markov code if for every periodic point x in $X_{\mathcal{C}}$ the indices $I_k, k \in \mathbb{Z}$, such that (2.1), (2.2), and (2.3) hold, are uniquely determined by x and can then be denoted by $I_k(x), k \in \mathbb{Z}$. If \mathcal{V} contains one element then one has a circular code (see e.g. [2]).

Generalizing the formula for the zeta function of $X_{\mathcal{C}}$, where \mathcal{C} is a circular code, Keller [12] has proved a formula for the zeta function of $X_{\mathcal{C}}$, where \mathcal{C} is a circular Markov code. For completeness we reproduce here Keller's proof for the special case that we have in mind.

Given a circular Markov code (\mathcal{C}, s, r) denote by $\mathcal{C}(u, w)$ the set of words $c \in \mathcal{C}$ such that $s(c) = u, r(c) = w, u, w \in \mathcal{V}$. Set

$$g_{\mathcal{C}(u, v), n} = \text{card}\{c \in \mathcal{C} : s(c) = u, r(c) = v, \ell(c) = n\},$$

and introduce the matrix

$$H^{(\mathcal{C})}(z) = (g_{\mathcal{C}(u, v), n}(z))_{u, v \in \mathcal{V}}.$$

Theorem 2.1 (Keller). *For a circular Markov code (\mathcal{C}, s, r) ,*

$$\zeta_{X_{\mathcal{C}}}(z) = \det(I - H^{(\mathcal{C})}(z))^{-1}.$$

Proof. Let $n \in \mathbb{N}$. Consider triples of the form $(j, c_1, c_1 \cdots c_k)$, where $k \in \mathbb{N}$, and where

$$\begin{aligned} c_l \in \mathcal{C}, \quad 1 \leq l \leq k, \quad s(c_1) = r(c_k), r(c_l) = s(c_{l+1}), \quad 1 \leq l \leq k, \\ \ell(c_1 \cdots c_k) = n - \ell(c_1), \quad j = \ell(c_1) \end{aligned}$$

To every point $x \in X_{\mathcal{C}}$ of period n one assigns a triple of this kind, where $k \in \mathbb{N}$ is determined by

$$n = I_k(x) - I_0(x), \quad j = -I_0(x),$$

and

$$c_l = x_{[I_{l-1}(x), I_l(x))}, \quad 1 \leq l < k.$$

Due to the circularity of (\mathcal{C}, r, s) this assignment is bijective.

Denote by $\gamma_i(f)$ the i -th coefficient of the power series of a function f . From

$$\begin{aligned} \gamma_j((H^k)_{u, w}) = \text{card}\{c_1 \cdots c_k : c_l \in \mathcal{C}, 1 \leq l \leq k, s(c_1) = u, r(c_k) = w, \\ r(c_l) = s(c_{l+1}), 1 \leq l \leq k, \ell(c_1 \cdots c_k) = j\}, \quad j, k \in \mathbb{Z}_+, \end{aligned}$$

one has

$$(\gamma_j(H)\gamma_{n-j}(H^k))_{u,w} = \text{card}\{c_1 \cdots c_k : c_l \in C, 1 \leq l \leq k, s(c_1) = u, r(c_k) = w, \\ r(c_l) = s(c_{l+1}), 1 \leq l \leq k, \ell(c_1) = j, \ell(c_1 \cdots c_k) = n - j\}, \quad 1 \leq j \leq n, \quad k \in \mathbb{Z}_+.$$

It follows that

$$\begin{aligned} \log \zeta_{X_C}(z) &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} j \text{ card}\{c_1 \cdots c_k : c_l \in C, 1 \leq l \leq k, r(c_l) = s(c_{l+1}), 1 \leq l \leq k, \\ &\quad s(c_1) = r(c_k), \ell(c_1) = j, \ell(c_1 \cdots c_k) = n\} \\ &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{1 \leq j \leq n} j \text{ trace} \left(\sum_{k \in \mathbb{Z}_+} \gamma_j(H) \gamma_{n-j}(H^k) \right) \\ &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{1 \leq j \leq n} j \text{ trace}(\gamma_j(H) \gamma_{n-j}((I - H)^{-1})) \\ &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{0 \leq j < n} (j + 1) \text{ trace}(\gamma_{j+1}(H) \gamma_{n-1-j}((I - H)^{-1})) \\ &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{0 \leq j < n} \text{ trace}(\gamma_j(H') \gamma_{n-1-j}(I - H)^{-1}) \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n} \text{ trace}(\gamma_{n-1}(H'(I - H)^{-1})z^n) \\ &= - \sum_{n \in \mathbb{N}} \text{ trace}(\gamma_n(\log(I - H))z^n) \\ &= - \text{ trace}(\log(I - H)). \end{aligned}$$

It is (see [9, Section 1.1.10]),

$$\text{ trace}(\log(I - H)) = \log \det(I - H),$$

and the theorem follows. \square

We state Keller's formula for the case of a circular code (see [29], [28], and references given in [4]) as a corollary.

Corollary 2.2. *For a circular code \mathcal{C}*

$$\zeta_{X_C}(z) = \frac{1}{1 - g_C(z)}.$$

Note that also the formula for the zeta function of a subshift of finite type in terms of a presenting polynomial matrix [3] is also a special case of Keller's formula.

Let G be a finite directed graph with adjacency matrix A_G . We introduce the Markov-Dyck codes $\mathcal{C}_v, v \in \mathcal{V}$ of words $c = (c_k)_{1 \leq k \leq K} \prod_{1 \leq k \leq K} c_k = P_v, \prod_{1 \leq j \leq J} c_k \neq P_v, 1 \leq J < K$. Standard methods of combinatorics (as for instance described in [7]) give

$$g_{\mathcal{C}_u}(z) = z^2 \sum_{v \in \mathcal{V}} \frac{A_G(u, v)}{1 - g_{\mathcal{C}_v}(z)}, \quad u \in \mathcal{V}$$

and by the implicit function theorem (1) has a unique solution (see for instance [23],[18]). Set

$$\mathcal{C} = \cup_{v \in \mathcal{V}} \mathcal{C}_v.$$

Also denote by \mathcal{C}^- the set of admissible words that are concatenations of an element (possibly empty) of \mathcal{P}^- with a word in \mathcal{C} and denote by \mathcal{C}^+ the set of admissible words that are concatenations of a word in \mathcal{C} and an element (possibly empty) of \mathcal{P}^+ . (\mathcal{C}^-, s, r) and (\mathcal{C}^+, s, r) are circular Markov codes. Denote by $D(A_G, z)$ the diagonal matrix with entries $g_{\mathcal{C}_v}(z), v \in \mathcal{V}$, and denote by $D^*(A_G, z)$ the diagonal matrix with entries $g_{\mathcal{C}_v}^*(z) = \frac{1}{1-g_{\mathcal{C}_v}(z)}, v \in \mathcal{V}$.

Theorem 2.3. *The zeta function of the Markov Dyck shift D_G is*

$$\begin{aligned}\zeta_{D_G}(z) &= \frac{1}{\det((I - D(A_G, z) - A_G z)(I - D^*(A_G, z)A_G z))} \\ &= \frac{\det(D^*(A_G, z))}{\det((I - D^*(A_G, z)A_G z))^2}\end{aligned}$$

Proof. Since $\sum_{k \in \mathbb{Z}_+} A_G^k z^k = (I - A_G z)^{-1}$ one has

$$H^{(\mathcal{C}^+)}(z) = D(A_G, z)(I - A_G z)^{-1},$$

and $H^{(\mathcal{C}^-)}(z)$ is the adjoint of $H^{(\mathcal{C}^+)}(z)$. Applying Proposition 2.1 and collecting all contributions to the zeta function, one has

$$\begin{aligned}\zeta_{D_G}(z) &= \left(\prod_{u \in \mathcal{V}} g_{\mathcal{C}_u}^*(z)^{-1} \right) \det(I - A_G z)^{-2} \det(I - D(A_G, z)(I - A_G z)^{-1})^{-2} \\ &= \left(\prod_{u \in \mathcal{V}} g_{\mathcal{C}_u}^*(z)^{-1} \right) \det(D^*(A_G, z)^{-1} - A_G z)^{-2} \\ &= \det((D^*(A_G, z)^{-1} - A_G z)(I - D^*(A_G, z)A_G z))^{-1} \\ &= \frac{1}{\det((I - D(A_G, z) - A_G z)(I - D^*(A_G, z)A_G z))} \\ &= \frac{\det(D^*(A_G, z))}{\det(I - D^*(A_G, z)A_G z)^2}.\end{aligned}$$

□

Inserting into the formula for the case of the graph with one vertex and N -loops, the generating function

$$g_{\mathcal{C}_v}(z) = \frac{1 - \sqrt{1 - 4Nz^2}}{2}$$

one obtains again the zeta function of the Dyck shift D_N as

$$\zeta_{D_N}(z) = \frac{2(1 + \sqrt{1 - 4Nz^2})}{(1 - 2Nz + \sqrt{1 - 4Nz^2})^2}$$

(see [12]).

3. TOPOLOGICAL ENTROPY

Proposition 3.1. *For the Markov-Dyck shift D_G*

$$h(D_G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n(D_G).$$

Proof. A word $b \in \mathcal{L}_n(D_G)$, $n \in \mathbb{N}$, determines words $a^+(b) \in \mathcal{D}^+$, $a^-(b) \in \mathcal{D}^-$, by

$$\prod_{1 \leq m \leq n} b_m = a^+(b)a^-(b),$$

as well as indices $I^-(b), I^+(b)$, $1 \leq I^-(b), I^+(b) \leq n$, by

$$I^-(b) = \min\{i : \prod_{1 \leq j \leq i} b_j = a^+(b)\}, \quad I^+(b) = \max\{i : \prod_{i \leq j \leq n} b_j = a^-(b)\}.$$

Denote by \mathcal{K}_n the set of $b \in \mathcal{L}(D_G)$ of length $n \in \mathbb{N}$ such that $I^-(b) = 1$. Choose for $u, w \in \mathcal{V}$ a path $c(u, w)$ in G from u to w of shortest length $\lambda(u, w)$ and set

$$L = \max_{u, w \in \mathcal{V}} \lambda(u, w).$$

We define a mapping Ψ_n of $\mathcal{L}_n(D_G)$ into $\cup_{n \leq m \leq n+2L} \mathcal{K}_m$ by

$$\Psi_n(b) = b_{[I^-(b), n]} * c(b_n, b_{I^-(b)}) * a^- * c(b_1, b_{I^-(b)}), \quad a^+(b) = a^+, b \in \mathcal{L}_n(D_G), n \in \mathbb{N}.$$

A word in \mathcal{K}_m , $n \leq m \leq n+2L$, has at most $(n+2L)\text{card}\mathcal{V}$ inverse images under the mapping Ψ_n , therefore

$$\text{card } \mathcal{L}_n(D_G) \leq (n+2L) \sum_{n \leq m \leq n+2L} |\mathcal{K}_m|. \quad (3.1)$$

Every word in \mathcal{K}_m , $n \leq m \leq n+2L$, determines a periodic point in D_G and (3.1) implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{L}_n(D_G)| \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n(D_G).$$

□

Corollary 3.2. *For the Markov-Dyck shift D_G , the topological entropy $h(D_G)$ is the minimum positive solution of the equation:*

$$\det(I - D^*(A_G, z)A_G z) = 0.$$

For $v \in \mathcal{V}$ let X_v denote the subsystem of the Markov-Dyck shift D_G that is obtained by excluding the words

$$e(-)e(+), \quad e \in E, \quad r(e) \in \mathcal{V} \setminus \{v\}$$

and the words

$$f(+)g(-), \quad g, f \in E, \quad s(f) \in \mathcal{V} \setminus \{v\}.$$

We will estimate the asymptotic growth rate of the periodic points of X_v which, by a proof that is similar to the proof of Proposition 3.1, is actually equal to the topological entropy of X_v . In this way, we will also obtain estimates of the topological entropy of the Markov-Dyck shifts.

For $v \in \mathcal{V}$ denote by \mathcal{D}_v the circular code of elementary Markov-Dyck words that start and end at v , and denote by \mathcal{C}_v the circular code of paths in G that start at v and end at v when they return for the first time to v . ρ denotes the inverse of the Perron eigenvalue of A_G . We denote by $p(z)$ the determinant of the matrix $I - A_G z$ and by $p_v(z)$ the determinant of the matrix $I - A_G z$ with the v -th row and the v -th column deleted, $v \in \mathcal{V}$. We set

$$q_v = \frac{\rho}{p_v}, \quad v \in \mathcal{V}.$$

Proposition 3.3.

$$g_{\mathcal{D}_v^* * \mathcal{C}_v^*}(z) = \frac{1 - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}}{2(1 - g_{\mathcal{C}_v}(z))}, \quad v \in \mathcal{V}.$$

Proof. One has $g_{\mathcal{D}_v^* * \mathcal{C}_v^*} = g_{\mathcal{D}_v} g_{\mathcal{C}_v^*}$ and $g_{\mathcal{D}_v}$ satisfies the equation

$$g_{\mathcal{D}_v}(z) = \frac{g_{\mathcal{C}_v}(z^2)}{1 - g_{\mathcal{D}_v}(z)}.$$

It follows that

$$g_{\mathcal{D}_v}(z) = \frac{1}{2}(1 - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}),$$

which yields the proposition. \square

Theorem 3.4. *Let $v \in \mathcal{V}$ be such that*

$$q_v(\rho^2) > \frac{3}{4}. \quad (3.2)$$

Then

$$h(X_v) > -\log \rho + \frac{p_v(\rho)[q_v(\rho^2) - \frac{3}{4} - \frac{1}{2}\sqrt{q_v(\rho^2) - \frac{3}{4}}]}{\rho p'(\rho)[\rho + \sqrt{q_v(\rho^2) - \frac{3}{4}}]}. \quad (3.3)$$

Proof. Corollary 2.2 and Proposition 3.3 imply that $h(X_v)$ is equal to $-\log \kappa$, where κ is the solution of the equation

$$1 = 2g_{\mathcal{C}_v}(z) - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}, \quad 0 < z < \rho,$$

or, equivalently, of the equation

$$\frac{1}{2} = q_v(z) + \sqrt{q_v(z^2) - \frac{3}{4}}, \quad 0 < z < \rho.$$

One has the estimate

$$\rho - \kappa > \frac{p_v(\rho)[q_v(\rho^2) - \frac{3}{4} - \frac{1}{2}\sqrt{q_v(\rho^2) - \frac{3}{4}}]}{p'(\rho)[\rho + \sqrt{q_v(\rho^2) - \frac{3}{4}}]}$$

and (3.3) follows. \square

Theorem 3.5. *Let $v \in \mathcal{V}$ be such that*

$$q_v(\rho^2) < \frac{3}{4}. \quad (3.4)$$

Then

$$h(X_v) > -\log \rho + \frac{p_v(\rho)}{2\rho^2 p'(\rho)}[q_v(\rho^2) - \frac{3}{4}] \quad (3.5)$$

Proof. Corollary 2.2 and Proposition 3.3 and (3.4) imply that $h(X_v)$ is greater than or equal to $-\log \kappa$, where κ is the solution of the equation

$$1 - q_v(z^2) = \frac{1}{4}, \quad 0 < z < \rho.$$

One has the estimate

$$\rho - \kappa > \frac{p_v(\rho)}{2p'(\rho)}[q_v(\rho^2) - \frac{3}{4}]$$

and (3.4) follows. \square

Lemma 3.6. *Let*

$$\rho < \frac{1}{4}. \quad (3.6)$$

Then there is a $v \in \mathcal{V}$ such that

$$q_v(\rho^2) > \frac{3}{4}.$$

Proof. Assume the contrary. Then

$$\frac{3}{4} \text{card} \mathcal{V} \leq \sum_{v \in \mathcal{V}} \frac{p_v(\rho^2)}{p(\rho^2)} = \text{trace}(I - A_{\rho^2}^{-1}) \leq \frac{1}{1 - \rho} \text{card} \mathcal{V},$$

contradicting (3.6). \square

Theorems 3.4 and 3.5 have as a corollary an estimate for the topological entropy of the Markov-Dyck shifts. We state here a corollary of Theorem 3.4.

Corollary 3.7. *Let*

$$\rho < \frac{1}{4}.$$

Then

$$h(D_G) > -\log \rho + \max_{\{v \in \mathcal{V}: q_v(\rho^2) > \frac{3}{4}\}} \frac{p_v(\rho)[q_v(\rho^2) - \frac{3}{4} - \frac{1}{2}\sqrt{q_v(\rho^2) - \frac{3}{4}}]}{\rho p'(\rho)[\rho + \sqrt{q_v(\rho^2) - \frac{3}{4}}]}.$$

Proof. The corollary follows from Theorem 3.4 by means of Lemma 3.6. \square

4. A CLASS OF EXAMPLES

We consider first the Fibonacci-Dyck shift D_F that is produced by the directed graph (Figure 1) with adjacency matrix $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Here

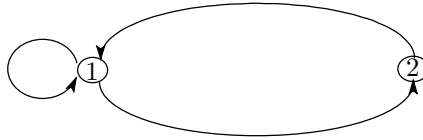


FIGURE 1

$$gc_1(z) = (gc_1^*(z) + gc_2^*(z))z^2, \quad (4.1)$$

$$gc_2(z) = gc_1^*(z)z^2, \quad (4.2)$$

where

$$gc_1^* = \frac{1}{1 - gc_1}, \quad gc_2^* = \frac{1}{1 - gc_2}, \quad (4.3)$$

or

$$gc_1 = 1 - \frac{1}{gc_1^*}, \quad gc_2 = 1 - \frac{1}{gc_2^*}. \quad (4.4)$$

From (4.1), (4.3) and (4.4)

$$g_{C_1^*}(z) = 1 + g_{C_1^*}(z)(g_{C_1^*}(z) + g_{C_2^*}(z))z^2, \quad (4.5)$$

and from (4.2) and (4.3)

$$g_{C_2^*}(z) = 1 + g_{C_1^*}(z)g_{C_2^*}(z)z^2. \quad (4.6)$$

From (4.2), (4.4) and (4.5)

$$g_{C_2^*}(z)^3 z^2 - g_{C_2^*}(z) + 1 = 0, \quad (4.7)$$

and from (4.6) and (4.7)

$$g_{C_1^*} = g_{C_2^*}^2. \quad (4.8)$$

From (4.2) and (4.3)

$$\det(I - Fz - D(F, z)) = \frac{z}{g_{C_2}}(g_{C_2}(z)^2 - (2z + 1)g_{C_2}(z) + z). \quad (4.9)$$

Setting $\xi(z) = g_{C_2^*}(z)z$ one has from (4.7)

$$\xi(z)^3 - \xi(z) + z = 0 \quad (4.10)$$

and from (4.9) and (4.10)

$$\det(I - Fz - D(F, z)) = -\frac{z^2}{\xi(z)^2}(2\xi(z)^2 + \xi(z) - 1). \quad (4.11)$$

By Theorem 2.3 and by (4.4) and (4.11)

$$\zeta_{D_F}(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2}, \quad (4.12)$$

where one identifies ξ as the solution of equation (4.10) vanishing at the origin that is given by

$$\xi(z) = \frac{2}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z\right), \quad 0 \leq z \leq \frac{2}{3\sqrt{3}}. \quad (4.13)$$

By Theorem 3.1 and by (4.12) the topological entropy of the Fibonacci-Dyck shift is equal to the negative logarithm of the solution of

$$2\xi(z)^2 + \xi(z) - 1 = 0.$$

By (4.10) (or by (4.13)),

$$h(D_F) = 3 \log 2 - \log 3. \quad (4.14)$$

We turn to the Markov-Dyck shift that is produced by the directed graph with adjacency matrix $F(a, b, c) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, $a, b, c \in \mathbb{N}$. Here

$$g_{C_1}(z) = (ag_{C_1^*}(z) + bg_{C_2^*}(z))z^2, \quad (4.15)$$

$$g_{C_2}(z) = cg_{C_1^*}(z)z^2 \quad (4.16)$$

and one has from (4.16) that

$$g_{C_1}(z) = 1 - \frac{cz^2}{g_{C_2}(z)}. \quad (4.17)$$

From (4.15) and (4.17)

$$ag_{C_2}(z)^3 - (a+c)g_{C_2}(z)^2 + c(1+(c-b)z^2)g_{C_2}(z) - c^2z^2 = 0. \quad (4.18)$$

From (4.17)

$$\det(I - F(a, b, c)z - D(F(a, b, c), z)) = \frac{z}{g_{C_2}}(ag_{C_2}(z)^2 - (a+c(1+b)z)g_{C_2}(z) + cz). \quad (4.19)$$

Theorem 2.3 and (4.17) and (4.19) give

$$\zeta_{D_{F(a,b,c)}}(z) = \frac{cg_{C_2}(z)(1-g_{C_2}(z))}{(ag_{C_2}(z)^2 - (a+c(1+b)z)g_{C_2}(z) + cz)^2}. \quad (4.20)$$

Setting

$$\begin{aligned} \mu(z) &= (c-a)^2 + ac - 3ac(c-b)z^2, \\ \nu(z) &= 2(a+c)^3 - 9ac(a+b) - (c-b+27a^2c^2)z^2, \end{aligned}$$

one identifies $g_{C_2}(z)$ as the solution of (4.18) that vanishes at the origin

$$g_{C_2}(z) = -\frac{a+c}{3a} + \frac{2}{3a}\sqrt{\mu(z)}\cos\left(\frac{1}{3}(2\pi + \arccos \frac{\nu(z)}{\mu(z)\sqrt{\mu(z)}})\right)$$

(For the case $a=b=c=1$ compare here (4.13)).

We determine the topological entropy of $D_{F(a,b,c)}$, $a, b, c \in \mathbb{N}$. Set

$$\begin{aligned} P_{a,b,c}(z) &= (1+c)[a(b-c) - c(1+b)^2]z^3 \\ &\quad + (c[(1+b)(1+c) - 2ab] + a(1+a-b))z^2 \\ &\quad + (bc - a - (1+a)(a-c))z + a - c. \end{aligned}$$

Theorem 4.1.

- (a) $h(D_{F(a,b,c)})$ is equal to the negative logarithm of the smallest positive solution of $P_{a,b,c}(z) = 0$, $a, b, c \in \mathbb{N}$.
- (b) $h(D_{F(a,b,a+b)}) = -\log(1+a+b)$, $a, b \in \mathbb{N}$.

Proof. Let $z > 0$ be such that the equations

$$ay^2 - (a+c(1+b)z)y + cz = 0 \quad (4.21)$$

and

$$ay^3 - (a+c)y^2 + c(1+(c-b)z^2)y - c^2z^2 = 0 \quad (4.22)$$

have a common solution y . Then y also solves the equation

$$(1 - (1+b)z)y^2 - (1-z+(c-b)z^2)y + cz^2 = 0 \quad (4.23)$$

and, as is seen from (4.21) and (4.23), it also solves the equation

$$(1 - (1+a+b)z)y = 1 - (1+a)z - b(1+c)z^2. \quad (4.24)$$

From (4.21) and (4.24)

$$\begin{aligned} bzP_{a,b,c}(z) &= (1 - (1+a)z - b(1+c)z^2) \\ &\quad \{a(1 - (1+a)z - b(1+c)z^2) - (1 - (1+a+b)z)(a+c(1+b)z)\} \\ &\quad + cz(1 - (1+a+b)z)^2 = 0. \end{aligned} \quad (4.25)$$

This shows that for every $z > 0$ such that equations (4.21) and (4.22) have a common solution, $P_{a,b,c}(z) = 0$.

Equation (4.23) is a multiple of equation (4.21) precisely if $c = a + b$ and $z = \frac{1}{1+a+b}$, and from this one sees, consulting (4.24), that both solutions of equation (4.21) are also solutions of equation (4.22) precisely if $c = a + b$ and $z = \frac{1}{1+a+b}$. Moreover, as is seen from (4.25), $P_{a,b,c}(z) = 0$ has the root $\frac{1}{1+a+b}$ if and only if $c = a + b$.

For the case that $c \neq a + b$, let $z > 0$,

$$P_{a,b,c}(z) = 0, \quad (4.26)$$

and reverse the argument, setting

$$y = \frac{1 - (1+a)z - b(1+c)z^2}{1 - (1+a+b)z}. \quad (4.27)$$

Consult then (4.24) and find from (4.26) that y as given by (4.27) solves equation (4.21) and therefore also equations (4.23) and (4.22). Apply now Theorem 2.3 in conjunction with Theorem 2.1 together with (4.18) and (4.20) to prove part (a) of the theorem for the case $c \neq a + b$.

Consider the case that $c = a + b$. One checks that $\frac{1}{1+a+b}$ is the unique positive root of $P_{a,b,a+b}(z) = 0$. It has already been shown that $P_{a,b,a+b}(z) = 0$ for every $z > 0$ such that equations (4.21) and (4.22), or, in this case, the equations

$$ay^2 - (a + (a+b)(1+b)z)y + (a+b)z = 0 \quad (4.28)$$

and

$$ay^3 - (2a+b)y^2 + (a+b)(1+az^2)y - (a+b)^2z^2 = 0$$

have a common solution. One checks that for $z = \frac{1}{1+a+b}$ a root of (4.28), in fact the smaller one, is equal to $g_{C_2}(\frac{1}{1+a+b})$. Apply now again Theorem 2.3 in conjunction with Theorem 2.1 together with (4.18) and (4.20) to prove part (a) of the theorem for the case that $c = a + b$ and also part (b). \square

The corollary reconfirms (4.14).

Corollary 4.2.

$$h(D_{F(a,1,a)}) = \log(a+1) - \log(a+2) + \log(a+3), \quad a \in \mathbb{N}$$

Determining the topological entropy of $D_{F(a,b,a)}$, $a, b \in \mathbb{N}$ and of $D_{F(a,b,c)}$ where $a, b, c \in \mathbb{N}$ solve $a(b-c) - c(1+b)^2 = 0$, reduces to solving a quadratic equation. In the various further cases expressions for the topological entropy are obtained from formulas of Tartaglia and Vieta.

REFERENCES

- [1] G. ABRAMS AND G. ARANDA-PINO, *The Leavitt path algebra of a graph*, J. Algebra **293**(2005), pp. 319–334.
- [2] J. BERSTEL AND D. PERRIN, *Theory of codes*, Academic Press, London (1985).
- [3] M. BOYLE, *Symbolic dynamics and matrices*, Combinatorial and Graph-theoretical Problems in Linear Algebra, IMA Volumes in Mathematics and its Applications **50**(1993), pp. 1–38.
- [4] M. BOYLE, J. BUZZI AND R. GOMEZ, *Almost isomorphism for countable state Markov shifts*, J. reine angew. Math. **592**(2006), pp. 23–47.
- [5] J. CUNTZ, *Simple C^* -algebras generated by isometries*, Commun. Math. Phys. **57**(1977), pp. 173–185.

- [6] J. CUNTZ AND W. KRIEGER, *A class of C^* -algebras and topological Markov chains*, *Inventiones Math.* **56**(1980), pp. 251–268.
- [7] M. DELEST, *Algebraic languages: a bridge between combinatorics and computer science*, *Dimacs* **24**(1996), pp. 71–87.
- [8] M. ENOMOTO AND Y. WATATANI, *A graph theory for C^* -algebras*, *Math. Japon.* **25**(1980), pp. 435–442.
- [9] I. P. GOULDEN AND D. M. JACKSON, *Combinatorial Enumeration*, John Wiley, New York, 1983.
- [10] T. HAMACHI, K. INOUE AND W. KRIEGER, *Subsystems of finite type and semigroup invariants of subshifts*, preprint.
- [11] K. INOUE, *The zeta function, periodic points and entropies of the Motzkin shift*, preprint, arXiv:math.DS/0602100.
- [12] G. KELLER, *Circular codes, loop counting, and zeta-functions*, *J. Combinatorial Theory* **56** (1991), pp. 75–83.
- [13] B. P. KITCHENS, *Symbolic dynamics*, Springer-Verlag, Berlin, Heidelberg and New York (1998).
- [14] W. KRIEGER, *On the uniqueness of the equilibrium state*, *Math. Systems Theory* **8** (1974), pp. 97–104.
- [15] W. KRIEGER, *On a syntactically defined invariant of symbolic dynamics*, *Ergodic Theory Dynam. Systems* **20**(2000), pp. 501–516.
- [16] W. KRIEGER, *On subshifts and semigroups*, *Bull. London Math.* **38** (2006), pp. 617–624.
- [17] W. KRIEGER AND K. MATSUMOTO, *A lambda-graph system for the Dyck shift and its K -groups*, *Doc. Math.* **8**(2003), pp. 79–96.
- [18] W. KUICH, *On the entropy of context-free languages*, *Information and Control* **16**(1970), pp. 173–200.
- [19] W. G. LEAVITT, *The module type of homomorphic images*, *Duke Math.J.* **32**(1965), pp. 305–31.
- [20] D. LIND AND B. MARCUS, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).
- [21] K. MATSUMOTO, *A simple purely infinite C^* -algebra associated with a lambda-graph system of Motzkin shift*, *Math. Z.* **248**(2004), pp. 369–394.
- [22] K. MATSUMOTO, *K -theoretic invariants and conformal measures of the Dyck subshift*, *Internat. J. Math.* **16** (2005), pp. 213–248.
- [23] K. MATSUMOTO, *Cuntz-Krieger algebras and a generalization of Catalan numbers*, preprint, arXiv:math.OA/0607517.
- [24] K. MATSUMOTO, *C^* -algebras arising from Dyck systems of topological Markov chains*, preprint, arXiv:math.OA/0607518.
- [25] K. MATSUMOTO, *K -theory for the simple C^* -algebra of the Fibonacci Dyck system*, preprint, arXiv:math.OA/0607519.
- [26] M. NIVAT AND J.-F. PERROT, *Une généralisation du monoïde bicyclique*, *C. R. Acad. Sc. Paris*, **271** (1970), pp. 824–827.
- [27] L. T. PATERSON, *Graph inverse semigroups, groupoids and their C^* -algebras*, *J. Operator Theory* **48**(2004), pp. 645–662.
- [28] D. PERRIN, *Algebraic combinatorics on words*, *Algebraic Combinatorics and Computer Science*, H.Crapo and G.-C.Rota, Eds. Springer 2001, pp. 391–430.
- [29] R. P. STANLEY, *Enumerative combinatorics I*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, (1986).

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